Kuwait University | Math 102 | 9 Nov, 2006
Dept of Math & Comp Sci | First Midterm | Duration: 90 min

Calculators, pagers and mobile telephones are not allowed.

1. (2+3 pts) Let

$$f(x) = \frac{1-e^x}{1+e^x}$$

(a) Show that f has an inverse.

(b) Find f^{-1} and state its domain.

2. (2 pts) Find the exact value of $\sin\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$.

3. (2 pts each) Find y' if

(a)

$$y = \sqrt{\frac{(\ln x)(\tan^{-1} x)}{2^x\sqrt{1+3x}}}$$

(b)

$$x^3 \sin^{-1}(e^{3x}) + \ln(\cosh y) = xy$$

4. (2 pts) Find the limit

$$\lim_{x\to 0} \frac{\cos 3x - \cos 5x}{x^2}$$

5. (3 pts) Given that $x = \ln(\csc\theta + \cot\theta)$, show that

$$\csc\theta = \cosh x$$

6. (3 pts each) Evaluate:

(a)

$$\int \frac{\sinh x \cosh x}{2+\sinh^2 x}\,dx$$

(b)

$$\int \frac{x^2}{\sqrt{1-x^6}}\,dx$$

(c)

$$\int \frac{2^x}{1+4^x}\,dx$$

Solutions

Q1. a) $f'(x) = \dfrac{-e^x(1+e^x)-(1-e^x)e^x}{(1+e^x)^2} = -\dfrac{2e^x}{(1+e^x)^2} < 0 \Rightarrow f$ one-to-one $\Rightarrow f$ has inverse.

b) $y = \dfrac{1-e^x}{1+e^x} \Rightarrow y(1+e^x) = 1-e^x \Rightarrow e^x(1+y) = 1-y$

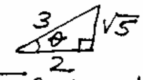
$\Rightarrow e^x = \dfrac{1-y}{1+y} \Rightarrow x = \ln\dfrac{1-y}{1+y} \Rightarrow f^{-1}(y) = \ln\dfrac{1-y}{1+y}$.

$\lim\limits_{x\to-\infty}\dfrac{1-e^x}{1+e^x} = \dfrac{1-0}{1+0} = 1$, $\lim\limits_{x\to\infty}\dfrac{1-e^x}{1+e^x} = \lim\limits_{x\to\infty}\dfrac{e^{-x}-1}{e^{-x}+1} = -1$

f decreasing & continuous $\Rightarrow$ Range $f = (-1,1) = \text{Dom } f^{-1}$.

Q2. $\sin\left(\cos^{-1}\left(\frac{2}{3}\right)\right) = \sin\theta$, where $\theta = \cos^{-1}\frac{2}{3} \in [0,\pi]$

$\Rightarrow \cos\theta = \frac{2}{3}$



$\Rightarrow \sin\theta = \frac{1}{3}\sqrt{9-4} = \frac{\sqrt{5}}{3}$.

Q3. a) $\ln y = \frac{1}{2}\left[\ln(\ln x) + \ln(\tan^{-1}x) - x\ln 2 - \frac{1}{2}\ln(1+3x)\right]$

$\Rightarrow \dfrac{y'}{y} = \dfrac{1}{2}\left[\dfrac{1}{x\ln x} + \dfrac{1}{(1+x^2)\tan^{-1}x} - \ln 2 - \dfrac{3}{2(1+3x)}\right]$

$\Rightarrow y' = \dfrac{1}{2}y[\cdots] = \dfrac{1}{2}\sqrt{\dfrac{(\ln x)\tan^{-1}x}{2^x\sqrt{1+3x}}}\,[\cdots]$.

b) $3x^2\sin^{-1}(e^{3x}) + x^3\dfrac{3e^{3x}}{\sqrt{1-e^{6x}}} + \dfrac{\sinh y}{\cosh y}y' = y + xy'$

$\Rightarrow y'(x-\tanh y) = 3x^2\left[\sin^{-1}(e^{3x}) + \dfrac{xe^{3x}}{\sqrt{1-e^{6x}}}\right] - y \Rightarrow y' = \cdots$

Q4. $\lim\limits_{x\to 0}\dfrac{\cos 3x - \cos 5x}{x^2} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0}\dfrac{-3\sin 3x + 5\sin 5x}{2x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0}\dfrac{-9\cos 3x + 25\cos 5x}{2}$

$= \frac{1}{2}(-9+25) = 8$.

Q.5. $e^x = \csc\theta + \cot\theta \Rightarrow e^{-x} = \dfrac{1}{e^x} = \dfrac{1}{\csc\theta + \cot\theta}$

$\Rightarrow \cosh x = \dfrac{1}{2}(e^x + e^{-x}) = \dfrac{1}{2}\left(\csc\theta + \cot\theta + \dfrac{1}{\csc\theta + \cot\theta}\right)$

$= \dfrac{(\csc\theta + \cot\theta)^2 + 1}{2(\csc\theta + \cot\theta)} = \dfrac{\csc^2\theta + 2\csc\theta\cot\theta + \cot^2\theta + 1}{2(\csc\theta + \cot\theta)}$

$= \dfrac{2\csc^2\theta + 2\csc\theta\cot\theta}{2(\csc\theta + \cot\theta)} = \csc\theta$.

Q.6. a) $u = 2 + \sinh^2 x$, $du = 2\sinh x\cosh x\,dx$ $\Rightarrow$ Integral $= \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln(2+\sinh^2 x) + C$

b) $u = x^3$, $du = 3x^2 dx$ $\Rightarrow$ Integral $= \frac{1}{3}\int\frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1}(x^3) + C$

c) $u = 2^x$, $du = 2^x(\ln 2)dx$ $\Rightarrow$ Integral $= \frac{1}{\ln 2}\int\frac{du}{1+u^2} = \frac{1}{\ln 2}\tan^{-1}(2^x) + C$.